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LETTER TO THE EDITOR

About the supersymmetric extension of the symplectic Faddeev–Jackiw quantization formalism

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Abstract. The key equations of the supersymmetric extension of the symplectic Faddeev–Jackiw quantization formalism are written in an alternative way. In this method the crucial problem is to compute the inverse of the symplectic supermatrix. We show how it can be easily given once the configuration space is defined.

As is well known, an alternative procedure concerning the quantization of systems described by first-order actions was given by Faddeev and Jackiw (FJ) [1]. In some cases the symplectic method is more economical than the usual Dirac formalism [2] for constrained Hamiltonian systems.

The FJ Lagrangian method is particularly useful when used in gauge models containing many dynamical variables involving several constraint equations.

The procedure developed by FJ is available for dynamical systems described by first-order Lagrange functions. This is not a crucial restriction, because any dynamical system can be written in first-order form, enlarging the configuration space by introducing suitable auxiliary fields.

Essentially, the FJ formalism where differential geometric techniques are used, shows how, by means of an unusual type of 'generalized brackets' on configuration space, a given system can be quantized along the ordinary prescriptions of the canonical quantization. In this picture, all the canonical information of a dynamical system is contained in the fundamental symplectic 2-form. The main feature of the symplectic formalism is that the classification of constrained or unconstrained systems is related to the singular or non-singular behaviour of the fundamental symplectic 2-form [3–5]. In contrast to the Dirac language, the classification of constraints in primary, secondary and so on, or in the first class and second class has no meaning. In the FJ symplectic treatment there are only constraints associated with gauge symmetries. When the FJ method is applied to gauge field theories in which true first-class constraint exist, the algorithm is unable to produce an invertible symplectic matrix. That is, the generalized brackets or commutators cannot be computed. In such a case a way of solving the situation is to break the gauge symmetries by adding gauge fixing terms in the action. Alternatively, one can solve the gauge constraints and rediagonalize the canonical 1-form.

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A useful point of view about the FJ formalism was recently given in [6]. In this paper it was shown how the FJ construction in non-Abelian systems leads to a singular symplectic matrix. The zero modes associated with the singular symplectic matrix generate a symmetry on the constraint surface, and so define the basic geometric structure of the model. Moreover, as can be seen clearly, the zero modes and the generators of gauge symmetries are closely related, leading to the correct transformation properties for the gauge fields.

On the other hand, an extension of the FJ formalism to include Grassmann dynamical variables can be found in [7]. The supersymmetric version developed in [7] has not often been used in supersymmetric systems. As we have shown in [8, 9], the algorithm turns out to be very powerful when it is applied to different supergravity models.

The purpose of the present letter is to write the key equations of the formalism in an alternative way that allows us to write general equations for the generalized graded commutators. Then, when particular models are considered, these equations can be easily computed.

As mentioned above, the FJ symplectic quantization method is based on an action only containing first-order time derivatives. The most general first-order action contains a Lagrangian density specified in terms of two arbitrary functionals $K_A(\mu^A)$ and $\boldsymbol{V}(\mu)$ which is given by

$$L(\mu_A, \dot{\mu}^A) = \dot{\mu}^A K_A(\mu^A) - \boldsymbol{V}(\mu). \tag{1}$$

The functionals, $K_A(\mu^A)$ are the components of the canonical 1-form $K(\mu) = K_A(\mu)\,\mathrm{d}\mu^A$ and the functional $\boldsymbol{V}(\mu)$ is the symplectic potential. Both are of even Grassmann parity and, therefore, $K_A(\mu)$ has Grassmann parity $|A|$, where the general compound index A runs over the different ranges of the complete set of variables. The set of dynamical field variables $\{\mu^A\}$ is given by the original set of fields plus a set of auxiliary fields necessary to bring the system into its first order form (1); consequently this set defines the extended configuration space.

The Euler–Lagrange equations of motion obtained from (1) are

$$\sum_B (-1)^{|B|} M_{AB}\dot{\mu}^B - \frac{\partial \boldsymbol{V}}{\partial \mu^A} = 0. \tag{2}$$

The elements of the simplectic supermatrix $M_{AB}(\mu)$ are the components of the symplectic 2-form $M(\mu) = \mathrm{d}K(\mu)$. The exterior derivative of the canonical 1-form $K(\mu)$ is written as the generalized curl constructed with functional derivatives and the components are, therefore, given by

$$M_{AB}(x, y) = \frac{\delta K_B(y)}{\delta \mu^A(x)} - (-1)^{|A||B|}\frac{\delta K_A(x)}{\delta \mu^B(y)}. \tag{3}$$

By definition, the Grassmann parity of the supermatrix M_{AB} is $(|A| + |B|)$ and the symmetry property is

$$M_{AB} = -(-1)^{|A||B|} M_{BA}. \tag{4}$$

When the simplectic supermatrix M_{AB} is non-singular, it defines the symplectic two-form characterizing the dynamical system described by (1). In such a case there is a unique quantity $(M^{AB})^{-1}$ with the property

$$\int \mathrm{d}z\, M_{AB}(x, z)(M^{BC})^{-1}(z, y) = \delta_A^C \delta(x, y) \tag{5}$$

$$\int \mathrm{d}z\, (M^{AB})^{-1}(x, z) M_{BC}(z, y) = \delta_C^A \delta(x, y). \tag{6}$$

From the equations of motion (2) we have

$$\dot{\mu}^A = (-1)^{|A|}(M^{AB})^{-1}\frac{\partial \boldsymbol{V}}{\partial \mu^B}. \tag{7}$$

As the symplectic potential is just the Hamiltonian of the system, the equation (7) is written as

$$\dot{\mu}^A = [\mu^A, \boldsymbol{V}\} = [\mu^A, \mu^B\}\frac{\partial \boldsymbol{V}}{\partial \mu^B} \tag{8}$$

where

$$[\mu^A(x), \mu^B(y)\} = (-1)^{|A|}(M^{AB})^{-1}(x, y) \tag{9}$$

are the generalized graded brackets defined in the supersymmetric extension of the FJ symplectic formalism.

Clearly from (4)–(6), the Grassmann parity of the supermatrix $(M^{AB})^{-1}$ is $(|A| + |B|)$ and so the symmetry property is given by

$$(M^{AB})^{-1} = -(-1)^{|A|+|B|+|A||B|}(M^{BA})^{-1}. \tag{10}$$

It is easy to show that the elements $(M^{AB})^{-1}$ of the inverse of the symplectic supermatrix M_{AB} correspond to the graded Dirac brackets of the theory. Transition to quantum theory is realized as usual by replacing classical fields by quantum field operators acting on some Hilbert space. Therefore, the predictions of both the FJ and Dirac methods are equivalent.

When the supermatrix M^{AB} is singular, the constraints appear as algebraic relations and they are necessary to maintain the consistency of the field equations of motion. In such a case, there exist m $(m < n)$ left (or right) zero modes $\boldsymbol{v}_{(\alpha)}$ $(\alpha = 1, \ldots, m,\ A = 1, \ldots, n)$ of the supermatrix M_{AB}, where each $\boldsymbol{v}_{(\alpha)}$ is a column vector with $n + m$ entries $v^A_{(\alpha)}$. So the zero modes verify the following equation:

$$\sum_A v^A_{(\alpha)} M_{AB} = 0. \tag{11}$$

From the equations of motion (2) we see that the quantities $\Omega_{(\alpha)}$ (of Grassmann parity $|\alpha|$), which are the true constraints in the FJ symplectic formalism, are given by

$$\Omega_{(\alpha)} = \int \mathrm{d}x\, v^i_{(\alpha)}(x, t)\frac{\delta}{\delta \varphi^i(x, t)}\int \mathrm{d}y\, \boldsymbol{V}(y, t) = 0. \tag{12}$$

Consequently, after a given iterated step (for instance the first step), the Lagrangian density can be written in the form

$$L^{(1)} = \dot{\varphi}^i a_i(\varphi) + \dot{\xi}^\alpha \Omega_\alpha - \boldsymbol{V}^{(1)} \tag{13}$$

where the partition $\mu^A = (\varphi^i, \xi^\alpha)$ and $K_A = (a_i, \Omega_\alpha)$ has been made. So, the compound indices A, B run over the set $A = (i, \alpha)$ and $B = (j, \beta)$.

In these conditions, the simplectic supermatrix in compact notation is written as

$$M_{AB}(x, y) = \begin{pmatrix} \bar{M}_{ij} & \frac{\delta \Omega_\beta(y)}{\delta \varphi^i(x)} \\ -(-1)^{|\alpha||j|}\left(\frac{\delta \Omega_\alpha(x)}{\delta \varphi^j(y)}\right) & 0 \end{pmatrix}. \tag{14}$$

In equation (13) we have assumed that $\varphi^i(x)$ represents any field belonging to the original symplectic set. Therefore, the square sub-supermatrix $\bar{M}_{ij}$ of the supermatrix (14), constructed from the original symplectic set of field variables, is non-singular. The compact notation $\delta\Omega_\alpha/\delta\varphi^j$ represents a rectangular supermatrix.

The symplectic algorithm must be repeated until all the non-orthogonal zero modes have been eliminated.

In each iterative procedure the configuration space is enlarged and the symplectic supermatrix is modified. When no new constraints are obtained the iterative procedure is finished.

As was commented above, if the matrix remains even singular, as in gauge theories, gauge fixing terms breaking the gauge symmetries can be added to the Lagrangian, and the graded commutators are computed in such particular gauge.

We assume that the inverse of the simplectic supermatrix M_{AB} can be written as

$$(M^{AB})^{-1}(x,y) = \begin{pmatrix} A^{jk}(x,y) & B^{j\rho}(x,y) \\ C^{\beta k}(x,y) & G^{\beta\rho}(x,y) \end{pmatrix}. \tag{15}$$

Using expressions (14) and (15) in equations (5) and (6), after some algebra the equations we obtain are

$$B^{j\rho}(x,y) = -(-1)^{|j|+|\rho|+|j||\rho|}C^{\rho j}(y,x) = -\int \mathrm{d}z\,\mathrm{d}w(\bar{M}^{jk})^{-1}(x,w)\frac{\delta\Omega_\beta(z)}{\delta\varphi^k(w)}G^{\beta\rho}(z,y) \tag{16}$$

$$A^{ij}(x,y) = (\bar{M}^{ij})^{-1}(x,y) - (-1)^{|k|+|\beta|+|k||\beta|}\int \mathrm{d}z\,\mathrm{d}w\left(\int \mathrm{d}u\,(\bar{M}^{ik})^{-1}(x,u)\frac{\delta\Omega_\beta(z)}{\delta\varphi^k(u)}\right) \times\left(\int \mathrm{d}v\,(\bar{M}^{il})^{-1}(z,v)\frac{\delta\Omega_\alpha(w)}{\delta\varphi^l(v)}\right)G^{\alpha\beta}(w,y) \tag{17}$$

$$\int \mathrm{d}z\,\Omega_{\alpha\beta}(x,z)G^{\beta\rho}(z,y) = \delta_\alpha^\rho\delta(x-y) \tag{18}$$

where

$$\Omega_{\alpha\beta}(x,z) = (-1)^{|\alpha||j|}\int \mathrm{d}y\,\mathrm{d}v\frac{\delta\Omega_\alpha(x)}{\delta\varphi^j(y)}(\bar{M}^{ji})^{-1}(y,v)\frac{\delta\Omega_\beta(z)}{\delta\varphi^i(v)}. \tag{19}$$

Finally, from equations (9) and (17), the graded brackets can be computed and they are written as follows:

$$[\varphi_i(x),\varphi_j(y)\} = (-1)^{|i|}(\bar{M}^{ij})^{-1}(x,y) -(-1)^{|i|+|j|+|\beta|+|j||\beta|}\int \mathrm{d}z\,\mathrm{d}w\left(\int \mathrm{d}u\,(\bar{M}^{ik})^{-1}(x,u)\frac{\delta\Omega_\beta(z)}{\delta\varphi^k(u)}\right) \times\left(\int \mathrm{d}v\,(\bar{M}^{jl})^{-1}(z,v)\frac{\delta\Omega_\alpha(w)}{\delta\varphi^l(v)}\right)G^{\alpha\beta}(w,y). \tag{20}$$

From equation (18) we see that the supermatrix $G^{\alpha\beta}(x,y)$ is none other than the inverse of the supermatrix defined above, whose matrix elements are $\Omega_{\alpha\beta}(x,y)$. According to equation (19), this supermatrix is constructed by using the set of constraints arising from the symplectic formalism, and involving the inverse of the non-singular supermatrix $\bar{M}_{ij}$. This way of writing the equations clearly shows how all the fundamental quantities remain determined only by the inverse of the non-singular submatrix $\bar{M}_{ij}$ and the square supermatrix $G^{\alpha\beta}(x,y)$. The form of $\bar{M}_{ij}$ is, in general, very simple once a suitable choice of the simplectic variables in (13) has been done. So, in the explicit evaluation of any graded bracket the key supermatrix is $G^{\alpha\beta}(x,y)$. When such a matrix exists, equation (20) gives the generalized graded brackets among every pair of fields of the complete set.

Of course, the supermatrix whose matrix elements are given in (19) is different from that constructed by using the set of constraints provided by the Dirac formalism. Moreover, as in the FJ algorithm there is, in general, a minor number of constraints, so the correspondent supermatrix is of lower dimension and the algebraic manipulations are shortened.

Now, when the supermatrix $G^{\alpha\beta}(x, y)$ does not exist the algorithm must be continued. The eigenvectors correspondent to the zero modes of the supermatrix (14) are computed from the general expression

$$\boldsymbol{v}(x) = (v_i(x), v_\beta(x)) = \left(\int \mathrm{d}z\, \mathrm{d}y (-1)^{|\alpha||j|} v_\alpha(z) \frac{\delta\Omega_\alpha(z)}{\delta\varphi^j(y)} \bar{M}^{-1}_{ji}(y, x), v_\beta(x) \right) \tag{21}$$

where the components v_α verify

$$\int \mathrm{d}z\, v_\alpha(z) \Omega_{\alpha\beta}(z, x) = 0. \tag{22}$$

Consequently, from the homogeneous linear equation system (22), the independent zero modes v_α can be computed.

The expressions obtained above arising from the supersymmetric extension of the symplectic FJ formalism are being checked in the supersymmetric nonlinear sigma model including the supersymmetric Hopf term. This is an example of a constrained system that in the Dirac picture contains first- and second-class constraints. As is well known, the quantization of this model following the Dirac prescription is very hard. Some useful conclusions will be given in a further paper by confronting both approaches.

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